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NONLINEAR OBSERVABILITY AND MIXED-COORDINATE BEARING-ONLY SIGNAL PROCESSING

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ABSTRACT

A simple test to determine nonlinear system observability is presented here and applied to several examples. Most interesting is the bearing-only tracking example which shows the need for relative maneuvers. A mixed-coordinate system is compared favorably to the standard rectangular coordinates and the popular modified polar coordinates for accuracy with a continuous-discrete truncated second-order filter.

1. Introduction

For linear or linearized systems, the simple rank condition is used to test observability, but for general nonlinear systems there exists the possibility of multiple-valued connections of some states to the measurements which implies that satisfaction of rank condition is not enough.

Many authors [1] - [8] have derived the conditions by which a given system may be found observable or not. But, unfortunately, they are usually insufficient [1], [2], too complicated to apply in practice [3], or applicable for only special forms of nonlinear systems such as in [8], ratio condition in [4], or for linearized systems [6], [7].

Here, we introduce a new method which is very simple to apply in practical problems and provides, not only, the test of observability of the system, but also, identifies the unobservable states when the system is unobservable.

Verification of the effectiveness of this method is shown by the tracking of a manuevering target where only bearing information is extracted from the measurement. When no maneuvering exists, either in the target and/or in the measuring ownship, the system is unobservable, but when proper manuevering exists the system is observable for all of the considered three different coordinate systems - rectangular, modified polar, and mixed coordinates.

A specially mixed coordinate combination of rectangular and polar coordinate components is introduced. The most desirable feature of this coordinate system is that, when the measurement noise level is high, these coordinates show the least estimation errors in both target-speed and position tracking, compared with the other two cases.

In the Section 2 the problem of checking nonlinear system obervability is discussed. Two conditions, connectedness and univalence, are provided along with examples. Section 3 analyzes a mixed coordinate system of equations as

well as rectangular and modified-polar-coordinate systems. In Section 4 a continuous-discrete, second-order filter is developed for the bearing-only target (BOT) motion which is described by the three coordinate systems. Comparisons between the coordinates are made. The last section contains some conclusions and suggested further research.

2. Nonlinear System Observability

System observability is directly related to the state estimation problem. If the system is not observable then the measurement does not provide enough information for proper estimation of those states.

Consider the system dynamic equation

$$\dot{x}(t) = f(x(t), u(t), t), \tag{1}$$

where $f(\cdot)$ is an n-function, $x \in R^n$, $u \in R^r$. The measurement equation is

$$y(t) = h(x(t), t), \tag{2}$$

where $h(\cdot)$ is an m-function, $y \in R^m$.

First assume that $y(t)$ is differentiable up to (n-1)-th order and $u(t)$ up to (n-2)-th order with respect to t, respectively. Then define system observability as follows: A state $x_i(t_0)$ is <u>observable to t_0</u> if knowledge of the input $u(t)$ and the output $y(t)$, $t \in [t_0, t_1]$ is sufficient to determine $x_i(t_0)$ for finite t_1. If every state $x(t) \in R^n$ is observable on $[t_0, t_1]$, then the system is <u>completely observable</u> or said to be <u>system observable</u>.

By differentiation of (2) (assumed to be sufficiently smooth), and substitution of (1)

$$
\begin{aligned}
y &= h(x, t) \\
y' &= \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} \cdot \frac{\partial x}{\partial t} = h_t + h_x f, \\
&\triangleq h_1(x, u, t),
\end{aligned}
$$

$$y'' = \frac{\partial h_1}{\partial t} + \frac{\partial h_1}{\partial x} \cdot \frac{\partial t}{\partial x} + \frac{\partial h_1}{\partial u} \cdot \frac{\partial u}{\partial t} = h_{1t} + h_{1x} f + h_{1u} u',$$

$$\triangleq h_2(x, u, u', t),$$

$$\vdots$$

$$y^{(n-1)} = h_{(n-2)t} + h_{(n-2)x} f + h_{(n-2)u} u' + \cdots + h_{(n-2)u^{(n-3)}} u^{(n-2)},$$

$$\triangleq h_{n-1}(x, u, u', \ldots, u^{(n-2)}, t),$$

where the time variable t is suppressed for convenience.

Define an mn measurement vector Y by

$$Y = \begin{bmatrix} y \\ y' \\ y'' \\ \vdots \\ y^{(n-1)} \end{bmatrix},$$

and an mn function $H(\cdot)$ to be

$$H(\cdot) = \begin{bmatrix} h \\ h_1 \\ h_2 \\ \vdots \\ h_{n-1} \end{bmatrix}.$$

Then one obtains an mn functional relation in vector form

$$Y = H(x, v, t), \tag{3}$$

where $v(t)$ is a function of $u^{(i)}$, $i = 1, \ldots, n-2$, and $u^{(i)}(t)$ is a known time function. Superscript i refers to the i-th derivative in time. Again, it is assumed that $u(t)$ is sufficiently smooth.

A Simple Useful Result

The system (1), (2) is observable if the following two conditions are satisfied:

(1) Connectedness

Every state $x_i(t)$, $i = 1, 2, \ldots, n$, is connected to any element of $\underline{\underline{y}}$, i.e., equation (3) must constitute n independent functions with respect to $x(t)$ in terms of $\underline{\underline{y}}$ on the time interval $t \in [t_0, t_1]$.

(2) Univalence

Every state $x_i(t)$, $i = 1, 2, \ldots, n$ is uniquely determined (single value) in terms of y.

The reasoning is complete if one can show that the unique connection of every state $x_i(t)$, $i = 1, 2, \ldots, n$, to any element of y is equivalent to that every state is connected to the measurement $y(t)$.

Let us expand analytic $y(t)$ in a Taylor series for any $t \in [t_0, t_1]$ at t_0, so that

$$y(t) = y(t_0) + y'(t_0)(t - t_0) + \frac{1}{2} y''(t_0)(t - t_0)^2 + \ldots$$
$$+ \frac{1}{(n-1)!} y^{(n-1)}(t_0)(t - t_0)^{n-1} + r(t) \qquad (4)$$

Thus knowledge of the measurement trajectory $y(t)$ on $[t_0, t_1]$ is equivalent to knowing each coefficient in (4) and the remainder $r(t)$. Since the Taylor series expansion for an analytic function is unique, each coefficient $y^{(i)}(t_0)$, $i = 1, \ldots, n-1$, is also unique. This implies that every coefficient contains the same amount of information about the state as $y(t)$ at t_0. However, the coefficients of expansion (4) are exactly the elements of the measurement vector y. Thus any state $x_i(t)$ is observable if it is connected to any element of y (connectedness). But further, this connection must

be one to one. Suppose it is not, i.e., some state $x_i(t)$ has multiple-valued expressions as

$$x_i(t) = y_1, y_2, y_3, \ldots, y_i$$

where y_i, i = 1, 2, ..., j, may be any appropriate function of the elements of y. Then x_i cannot be uniquely determined from the measurement y(t) since all y_i will give the same initial condition $x_i(t_0)$. Consequently, this state is not observable and the system is unobservable (univalence).

The above two conditions, while convenient, can be modified in terms of more concrete mathematical tools as follows:

(1) The connectedness condition corresponds to the <u>existence</u> of an inverse of the function (not necessarily unique) (3). Thus the global inverse-function theorem [15] can be used which states that there exists an inverse functon G: $R^n \times R^r \rightarrow R^n$ of $H_n(\cdot)$, where H_n is any subset of $H(\cdot)$ consisting of n-functions such that

$$G(y, v) = x$$

if

$$\det J \neq 0$$

for all x, v, where J is the Jacobian of H_n. In other words, if det J = 0, then at least one state is not connected to y. So, any states, which make the det J = 0, are unobservable.

(2) Further if any state has multiple solutions, then by imposing some constraints about that state, one can choose a unique solution by the following condition, i.e., if either J is <u>positive definite</u> or <u>negative definite</u> for all $x \in R^n$, then $H_n(\cdot)$ is one-to-one, thus H_n has a unique solution. (Refer to [16] for the proof.)

The test of this result is demonstrated readily by examples. The simple rank test for linear systems (time invariant and time variant) follows immediately. Then consider the following nonlinear examples.

Example 1 [2]

$$\dot{x}_1 = x_2 x_3$$

$$\dot{x}_2 = -x_1 x_3,$$

$$\dot{x}_3 = 0,$$

$$y = x_1.$$

Thus,

$$y' = \dot{x}_1 = x_2 x_3,$$

$$y'' = \dot{x}_2 x_3 + x_2 \dot{x}_3 = -x_1 x_3^2.$$

From the last three equations

$$x_1 = y,$$

$$x_2 = \pm\, y' / \sqrt{\frac{y''}{-y}},$$

$$x_3 = \pm \sqrt{\frac{y''}{-y}}.$$

Thus the nonzero initial state satisfies the connectedness condition, and

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & x_3 & x_2 \\ -x_3^2 & 0 & -2x_1 x_3 \end{bmatrix},$$

$\det J = -2x_1 x_3^2 \neq 0$, means that initial states of the form $x_{10} \neq 0$, $x_{30} \neq 0$, actually, satisfy this condition. But x_2, x_3 have multiple solutions, thus the univalence condition is not satisfied. So, positive or negative definiteness of J is tested and obtained as either $\{x_{10} < 0,\ x_{30} > 0\}$, or $\{x_{10} > 0,\ x_{30} < 0\}$ can make x_2 and x_3 have unique solutions.

Example 2 [3]

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -2x_1 - 3x_2 - x_1^3 x_3,$$

$$\dot{x}_3 = -x_3 x_4,$$

$$\dot{x}_4 = 0,$$

$$y = x_1.$$

So,

$$y' = x_2,$$

$$y'' = -2x_1 - 3x_2 - x_1^3 x_3,$$

$$y''' = 6x_1 + 7x_2 + 3x_1^3 x_3 + x_1^3 x_3 x_4 - 3x_1^2 x_2 x_3.$$

Then,

$$x_1 = y,$$

$$x_2 = y',$$

$$x_3 = \frac{-(2y + 3y' + y'')}{y^3},$$

$$x_4 = \frac{-(2y' + 3y'' + y''')}{2y + 3y' + y''} + \frac{3y'}{y}.$$

Here, the univalence condition is satisfied, but

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 - 3x_1^2 x_3 & -3 & -x_1^3 & 0 \\ j_{41} & 7 - 3x_1^2 x_3 & j_{43} & x_1^3 x_3 \end{bmatrix},$$

where

$$j_{41} = 6(1 - x_1 x_2 x_3) + 3x_1^2 x_3 (3 + x_4),$$

$$j_{43} = x_1^3 (3 + x_4) - 3x_1^2 x_2,$$

det J = $-x_1^6 x_3 \neq 0$, implies $\{x_{10} \neq 0, x_{30} \neq 0\}$, makes all the states connected to y. Thus only such initial states make the system observable.

Obviously, the method has limitations with respect to the required smoothness and existence of the inverse functions. But on the plus side, it seems easy to apply to fairly complicated nonlinear systems, shows which states are obversable and shows the effect of control as depicted next.

3. Bearings Only Track (BOT)

Consider an object or target (T) and observer (0) configuration as in Figure 1. When T and/or 0 move with some velocity components v_{Tx}, v_{Ty} and v_{0x}, v_{0y}, relative coordinate x(t) and y(t) can be generated as

$$\begin{aligned} x(t) &= x_T(t) - x_0(t), \\ y(t) &= y_T(t) - y_0(t). \end{aligned} \tag{5}$$

Define state variables for rectangular coordinates by

$$\begin{aligned} x_1(t) &= x_T(t) - x_0(t) = x(t), \\ x_2(t) &= v_{Tx}(t) - v_{0x}(t) = v_x(t), \\ x_3(t) &= y_T(t) - y_0(t) = y(t), \\ x_4(t) &= v_{Ty}(t) - v_{0y}(t) = v_y(t). \end{aligned} \tag{6}$$

Then, the state equation has the following linear form

$$x(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ a_x(t) \\ 0 \\ a_y(t) \end{bmatrix} , \tag{7}$$

where a_x, a_y are acceleration components, and the bearing measurement equation is

$$y(t) = \tan^{-1}\left(\frac{x_1(t)}{x_3(t)}\right). \tag{8}$$

In <u>modified-polar coordinates</u>, four state variables are defined as follows:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} \beta(t) \\ \dot{r}(t)/r(t) \\ \dot{\beta}(t) \\ 1/r(t) \end{bmatrix}, \tag{9}$$

where β is bearing and r is range.

After differentiation and algebraic manipulation with rectangular acceleration components $a_x(t)$ and $a_y(t)$, the state equation can be expressed as

$$\dot{x}(t) = \begin{bmatrix} -2x_1x_2 + x_4(a_x\cos x_3 - a_y\sin x_3) \\ x_1^2 - x_2^2 + x_4(a_x\sin x_3 - a_y\cos x_3) \\ x_1 \\ -x_2x_4 \end{bmatrix}, \tag{10}$$

and the measurement equation is

$$y(t) = [0 \quad 0 \quad 1 \quad 0]x(t). \tag{11}$$

Next instead of using normalized state as in modified polar coordinates, define the states by a mix of polar coordinate components r, β and rectangular components v_x, v_y so that

$$x_1(t) = \beta(t),$$

$$x_2(t) = r(t), \tag{12}$$

$$x_3(t) = v_x(t),$$

$$x_4(t) = v_y(t).$$

Then, the state equation in the mixed coordinate system becomes

$$\dot{x}(t) = \begin{bmatrix} \dfrac{x_3 \cos x_1 - x_4 \sin x_1}{x_2} \\ x_3 \sin x1 + x_4 \cos x_1 \\ a_x \\ a_y \end{bmatrix} . \tag{13}$$

The measurement equation is

$$y(t) = [1 \quad 0 \quad 0 \quad 0]x(t) . \tag{14}$$

Next observability of the mixed coordinate system is checked according to the conditions derived earlier for the two cases where maneuvering exists, i.e., $a_x(t) \neq 0$ and/or $a_y(t) \neq 0$, and nonmaneuvering.

From (13) and (14) with $a_x(t) = 0$, $a_y(t) = a(t) \neq 0$ (i.e., manuevering exists only in the one direction) and by replacing lower-order derivatives of $y(t)$ to the higher-order derivatives recursively,

$$y = x_1, \tag{15}$$

$$y' = \frac{x_3 \cos y - x_4 \sin y}{x_2}, \tag{16}$$

$$y'' = \frac{-(a \sin y + 2y' x_4 \cos y + 2y' x_3 \sin y)}{x_2}, \tag{17}$$

$$y''' = -\frac{3ay' \cos y + x_3(3y'' \sin y + 2(y')^2 \cos y) + x_4(3y'' \cos y - 2(y')^2 \sin y)}{x_2}. \tag{18}$$

Then (15) - (18) shows that

$$x_1 = y, \tag{19}$$

$$x_2 = \frac{-2y'x_4 - a\cos y \cdot \sin y}{A}, \tag{20}$$

$$x_3 = \frac{(y''\sin y - 2(y')^2\cos y)x_4 - y'a\sin y}{A} \tag{21}$$

$$x_4 = \frac{a[\sin y'\cos y \cdot y''' - (3Ay'\cos y - y'\cos y)(3y''\sin y + 2(y')^2\cos y)]}{-2y'y''' + (y''\sin y - 2(y')^2\cos y)(3y''\sin y + 2(y')^2\cos y) - (3y''\cos y - 2(y')^2\sin y)A}, \tag{22}$$

where $A = y''\cos y + 2(y')^2)\sin y$.

From (22) it is clear that, if $a(t) \neq 0$ (manuevering exists), then x_4 is connected to the measurement vector Y, it is unique, and thus observable. This implies from (20), (21) that x_2 and x_3 are also uniquely connected. x_1 is already observable from (19). So the system is observable for all nonzero initial conditions in $x(t_0) \in R^4$.

But when $a(t) = 0$ (non-maneuvering), then while $x_4 = x_{40}$, (22) suggests that $x_4 = 0$; i.e., x_4 is not connected to Y and is unobservable. This causes, again from (20) and (21), that x_2 and x_3 are disconnected from Y, and thus unobservable also. Only x_1 is observable in this case. Thus the known result, i.e., bearing-only target tracking system is observable when maneuvering exists is proved using the proposed two observability conditions provided earlier.

After lengthy computation, the determinant of the Jacobian becomes

$$\det J = \frac{6[a(y''\sin y - (y')^2\cos y) + y'y''(x_3\sin y + x_4\cos y)]}{x_2^4} \tag{23}$$

Consequently, the BOT is unobservable (in x_2, x_3, x_4) with det J = 0 for the following cases (among others):

i) Infinite range, $x_2 = \infty$;

ii) zero heading rate and acceleration, $y' = y'' = 0$;

iii) $x_3 = x_4 = 0$, i.e., $\dot{x}_2 = 0$, with $a(t) = 0$ (parallel stationary movement, including tail chase);

iv) constant range with special heading such that $\tan\beta = \beta''/\beta'^2$.

In these cases, as well as certain others, more measurements are required.

Using similar procedures, it is easily checked that the system is observable only when manuevering exists for both rectangular and modified-polar coordinate systems as well.

While observability for stochastic systems is not exactly the same as for deterministic systems, the concept is still useful in the same context as is an observer relative to a filter. In any event, it must be realized that these are only models of the real process.

The Fisher information matrix J may be computed to determine the relative information in the observations from, the conditional expectation,

$$J_k = -E\left[\frac{\partial^2 \ell n p_k(y_k/x_k)}{\partial x_k^2}\right]\left[x_k\right] , \tag{24}$$

taken at the kth instant of time for the discrete representation. For this example with white Gaussian measurement noise, and no state noise, the information matrix traditionally is given by the inverse error covariance matrix associated with the filter algorithm. Computations of J_k recursively verify the relative lack of observability for the nonmaneuvering case.

In the stochastic case, however, the information matrix, degree of observability in the measurements and effectiveness of the filter algorithm depends

strongly on the coordinate system. Comparisons have been made between modified polar coordinates [10], relative coordinates [9], [11], range direction cosine [12] and modified spherical coordinates [13]. Here the above mixed coordinates are compared with rectangular and modified polar coordinates.

Of course, some error in the estimation is due to the nonlinear measurements and the finite filter approximation used.

4. Mixed Coordinate Simulation and Comparison

To observe the effect of observability in the nonlinear system and to compare the usefulness of the proposed mixed coordinate system with other coordinates, continuous system - discrete observation type, truncated second-order filter is studied. With the T, O configuration as in Figure 1, and with assumed manuevering

$$a_x(t) = 0$$

$$a_y(t) = -0.025\cos(0.005t),\ [m/s^2]. \tag{25}$$

A continuous-discrete filter is developed in two stages, i.e., at the measurement update stage observed data is processed according to the discrete form filter, and at the second stage, between observation, time propagation integrals of the first and second moments of state estimation are processed according to the continuous fashion [14]. To give emphasis on the bearing measurement noise effect, it is assumed that the dynamic equations are noise free. But initial states are assumed Gaussian with mean

$$x_0 = [\beta_0,\ r_0,\ v_{x0},\ v_{y0}]^T \text{ and}$$

variance σ^2_{x0}.

Other important parameters used are:

$T = 10$ sec (measurement interval),

$\Delta t = 1$ sec (time update interval),

$r(0) = 8000$ m (range at $t = 0$),

$v_{Tx} = 10$ m/s $\simeq 20$ kTs, $v_{Ty} = 0$,

$v_{0x} = 15$ m/s $\simeq 30$ dTs, $v_{0y} = 5\sin(0.005t)$,

The measurement noise sequence is also assumed Gaussian.

Figures 2, 3 shows the effect of observability in estimation accuracy for the three different coordinates. Figure 2 shows speed error and Figure 3 shows range error. Clearly errors do not converge to zero in any sense in this algorithm when system is not observable (non-maneuvering) in all of the three coordinates.

Figures 4 to 7 show the comparison of range errors in estimation according to the bearing measurement noise level increases. In both low and high noise levels, mixed coordinates produce the least absolute errors. As the noise increases, the difference in filtering accuracy becomes significant. In fact in the mixed coordinates, actual absolute error does not increase by much, which is the most desirable characteristic in the real applicational point of view. In rectangular coordinates, the error converges asymptotically to zero with longer convergence time than mixed coordinates. The most undesirable characteristics in this simulation are shown in modified polar coordinates. Errors occur which are always larger than the other two cases, and exhibit oscillation.

5. Conclusion

To check observability of nonlinear systems and to identify unobservable states, two conditions (connnectedness and univalence were introduced. Since this method does not require any complicated mathematical manipulations beyond the apparent inverse functions, its utility is exhibited by medium dimensional nonlinear systems. Special examples of the method are shown for the bearing-only target motion equation using so-called "mixed coordinates" where the state equations are described by mixing rectangular and polar coordinate components. The system is observable only when proper maneuvering exists. When target and/or observer do not maneuver some states are not observable for the BOT.

The effects of observability in the estimation problem is shown to be significant in this simulation. Estimation error converges to zero quite fast when the system is observable. On the other hand, it does not converge in any sense in the unobservable case.

Comparison of estimation accuracy between three different coordinate systems is made. Mixed coordinates show the most desirable feature even when the measurement noise level is quite high. However continued research is necessary on the stochastic nonlinear observability problem, and its effect on the nonlinear estimation of the state to provide more rigorous theoretical background for the superiority of mixed coordinate systems as well as other applications.

REFERENCES

[1] Y. M. Kostyukovskii, "Observability of nonlinear controlled system," Automation Remote Control, Vol. 9, pp. 1384-1396, 1968.

[2] U. Schoenwandt, "On observability of nonlinear systems," Proc. 2nd IFAC Symposium on Identif., Prague, Czechoslovakia, Jun 1970.

[3] E. W. Griffith, K. S. P. Kumar, "On the observability of nonlinear systems I," J. of Math. Ana. and Appl., Vol. 35, pp. 135-147, 1971.

[4] S. R. Kou, D. L. Elliott, T. J. Tarn, "Observability of nonlinear systems," Info. and Control, Vol. 22, pp. 89-99, 1973.

[5] J. M. Fitts, "On the observability of nonlinear systems with application to nonlinear regression analysis," Info. Sciences, Vol. 4, pp. 129-156, 1972.

[6] E. B. Lee, L. Marcus, Foundations of Optimal Control Theory, John Wiley and Sons, New York, 1967.

[7] M. Hwang, J. H. Seinfeld, "Observability of nonlinear systems," J. of Opt. Theory and Appl., Vol. 10, No. 2, pp. 67-77, 1972.

[8] P. Sen, M. R. Chidambara, "Observability of a class of nonlinear systems," IEEE Tr. Auto. Cont., Vol. AC-25, No. 6, pp. 1236-1237, Dec 1980.

[9] R. R. Tenny, R. S. Hebbert, N. R. Sandell, "A tracking filter for maneuvering sources," IEEE Tr. Auto. Cont., Vol. AC-22, pp. 246-251, Apr 1977.

[10] S. E. Hammel, V. J. Aidala, "Utilization of modified polar coordinates for bearing-only tracking," IEEE Tr. Auto. Cont., Vol. AC-28, No. 3, pp. 283-294, Mar 1983.

[11] H. Weiss, J. B. Moore, "Improved extended Kalman filter design for passive tracking," IEEE Tr. Auto. Cont., Vol. AC-25, pp. 807-811, Aug 1980.

[12] R. K. Mehra, "A comparison of several nonlinear filters for reentry vehicle tracking," IEEE Tr. Auto. Cont., Vol. AC-16, pp. 307-319, Aug 1971.

[13] P. L. Bongiovanni, R. E. Silva, "A three dimensional underwater tracking problem using canonical angle measurements," 17th Asilomar Conference on Circuits, Systems, and Computers, Sep 1983.

[14] P. S. Maybeck, Stochastic Models, Estimation and Control (Vol. 2), Academic Press, New York, 1982.

[15] R. S. Palais, "Natural operation of differential forms," Trans. Amer. Math. Soc., Vol. 92, pp. 125-141, Jul 1959.

[16] E. S. Kuh, I. N. Hajj, "Nonlinear circuit theory: resistive networks," Proc. of IEEE, Vol. 59, No. 3, pp. 340-355, Mar 1971.

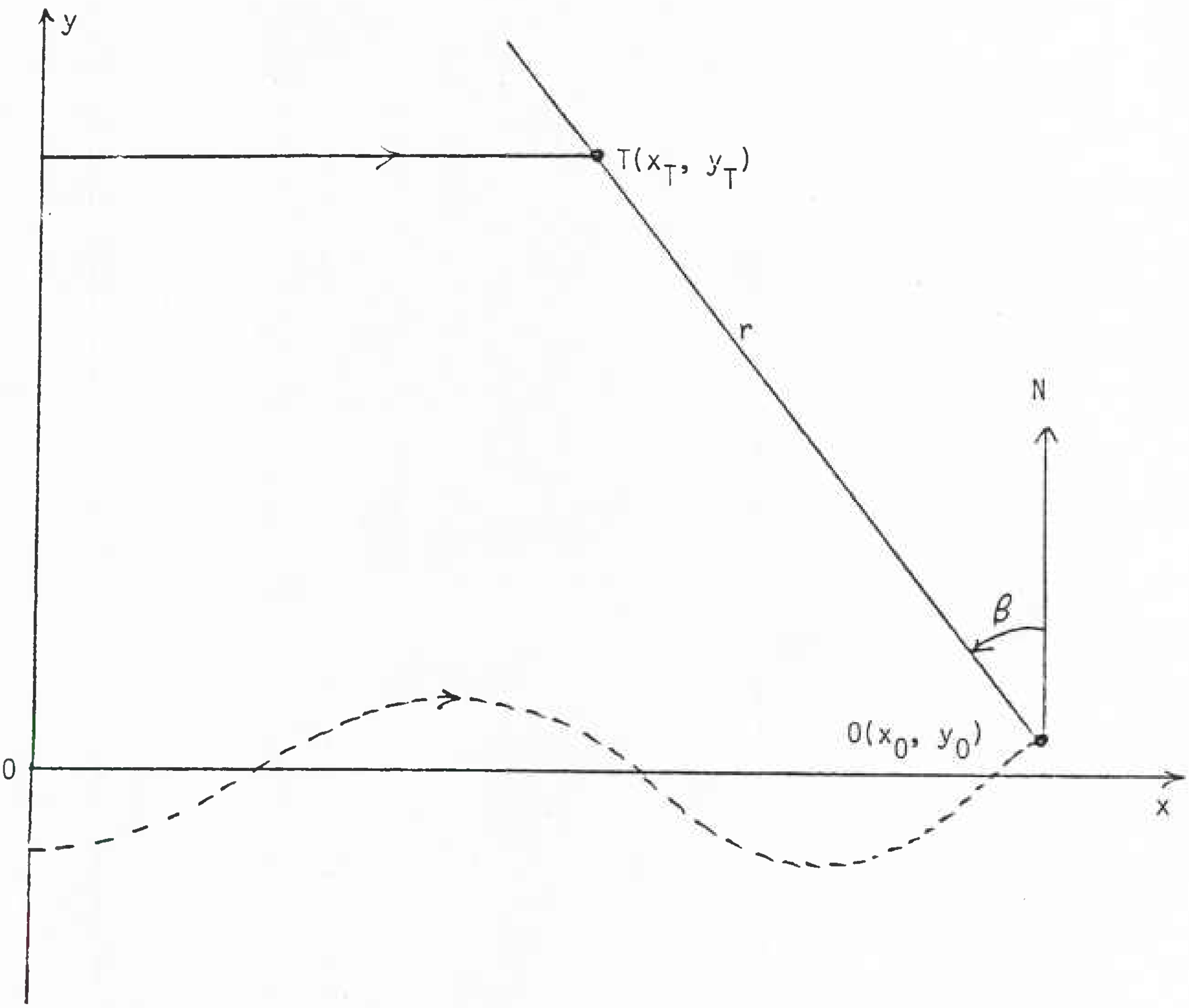
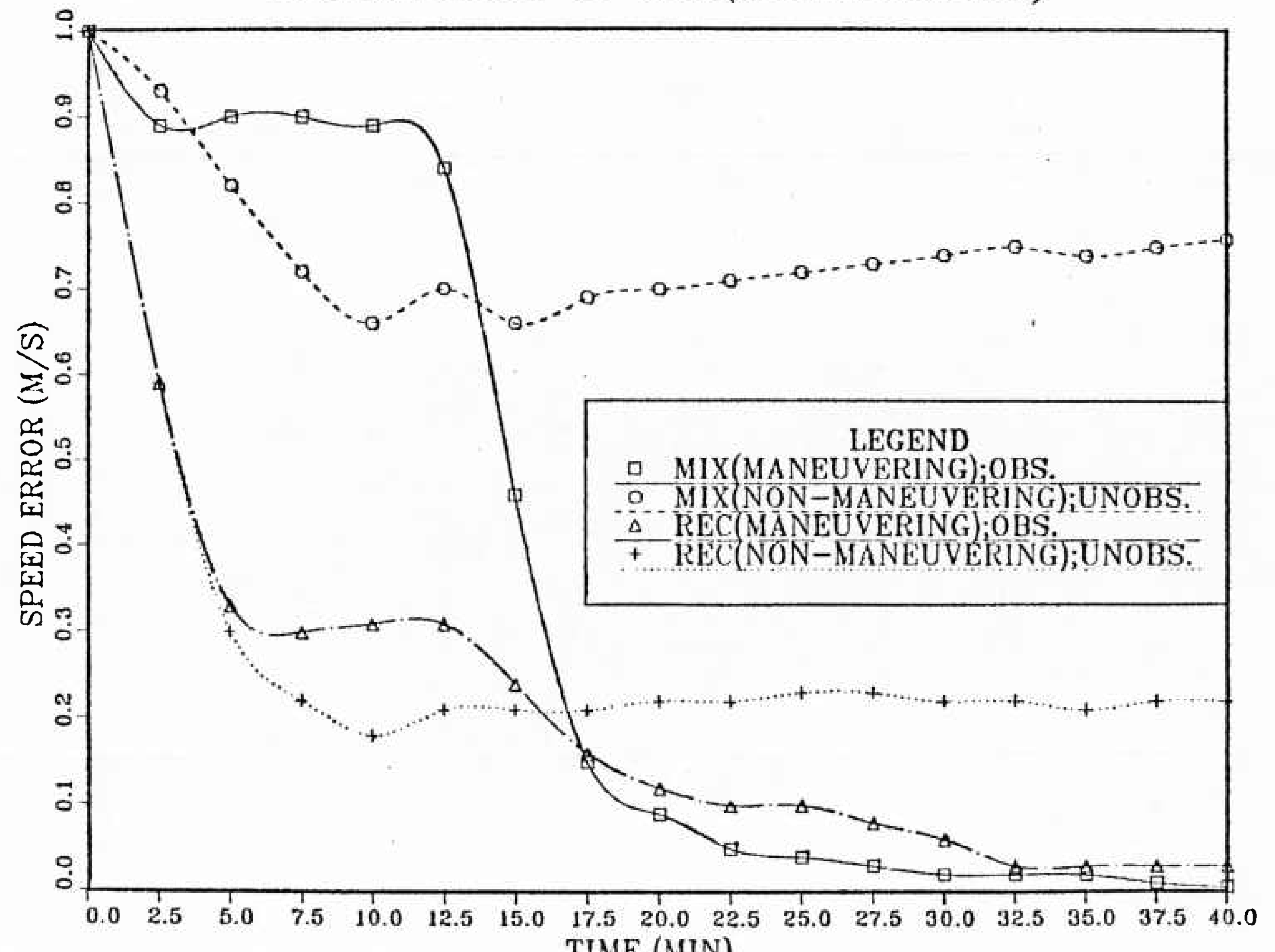


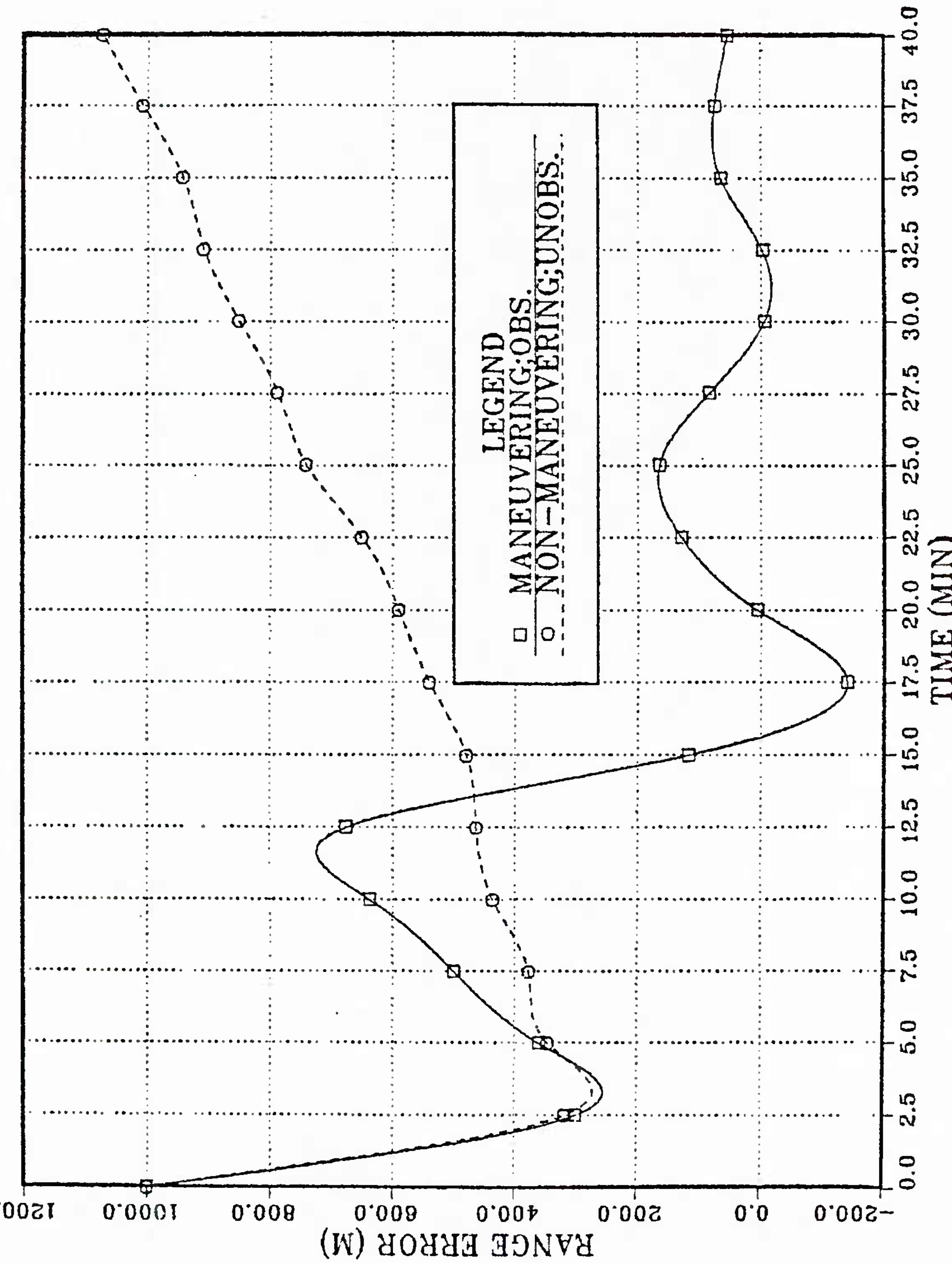
Figure 1. BOT Configuration.

FIG 2 ,EFFECT OF OBS(SPEED ERROR)



AVG OF 20 RUNS

FIG 3,EFFECT OF OBS(RANGE ERROR)



(A) MIXED-COORDINATE

(B) RECTANGULAR COORDINATE

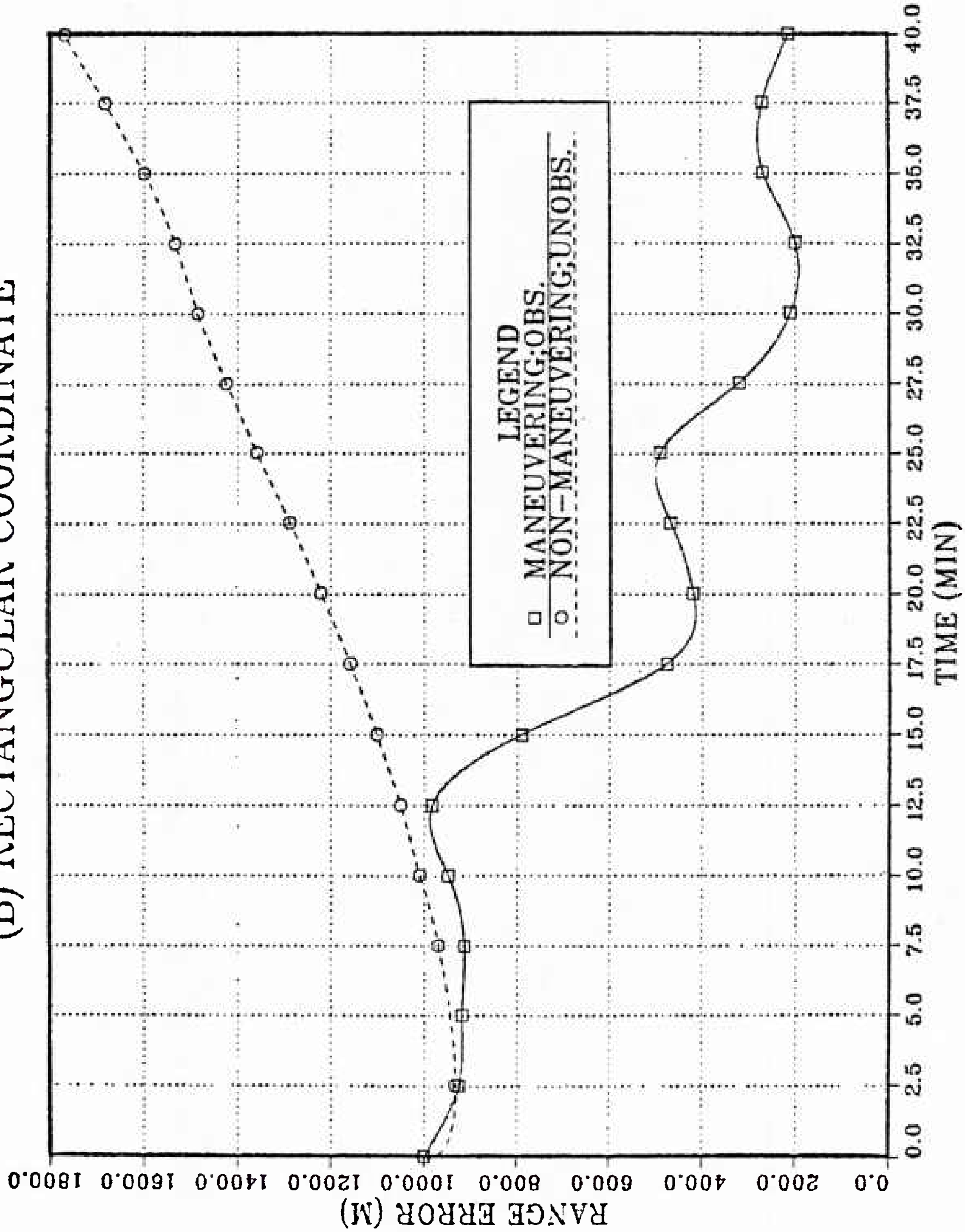
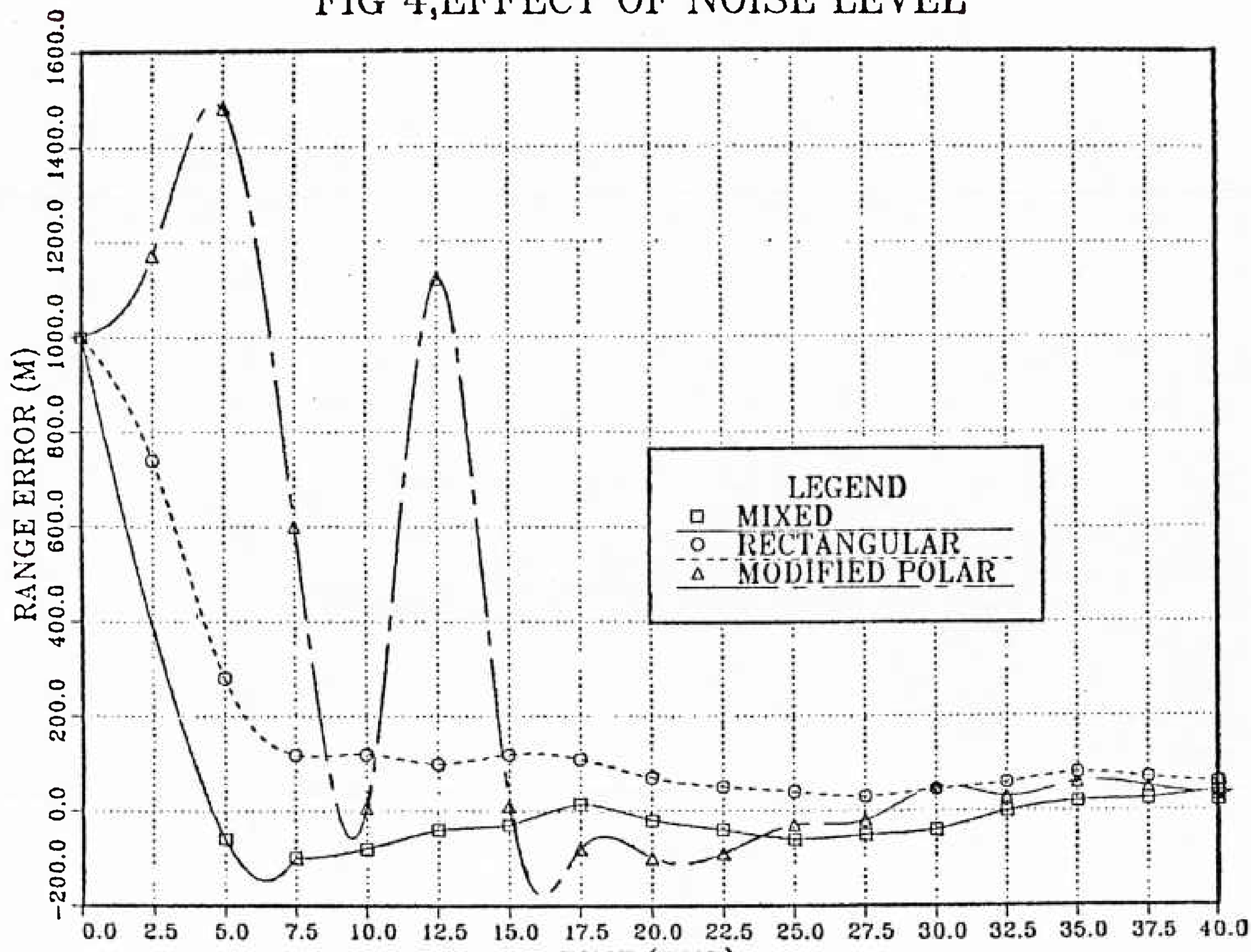
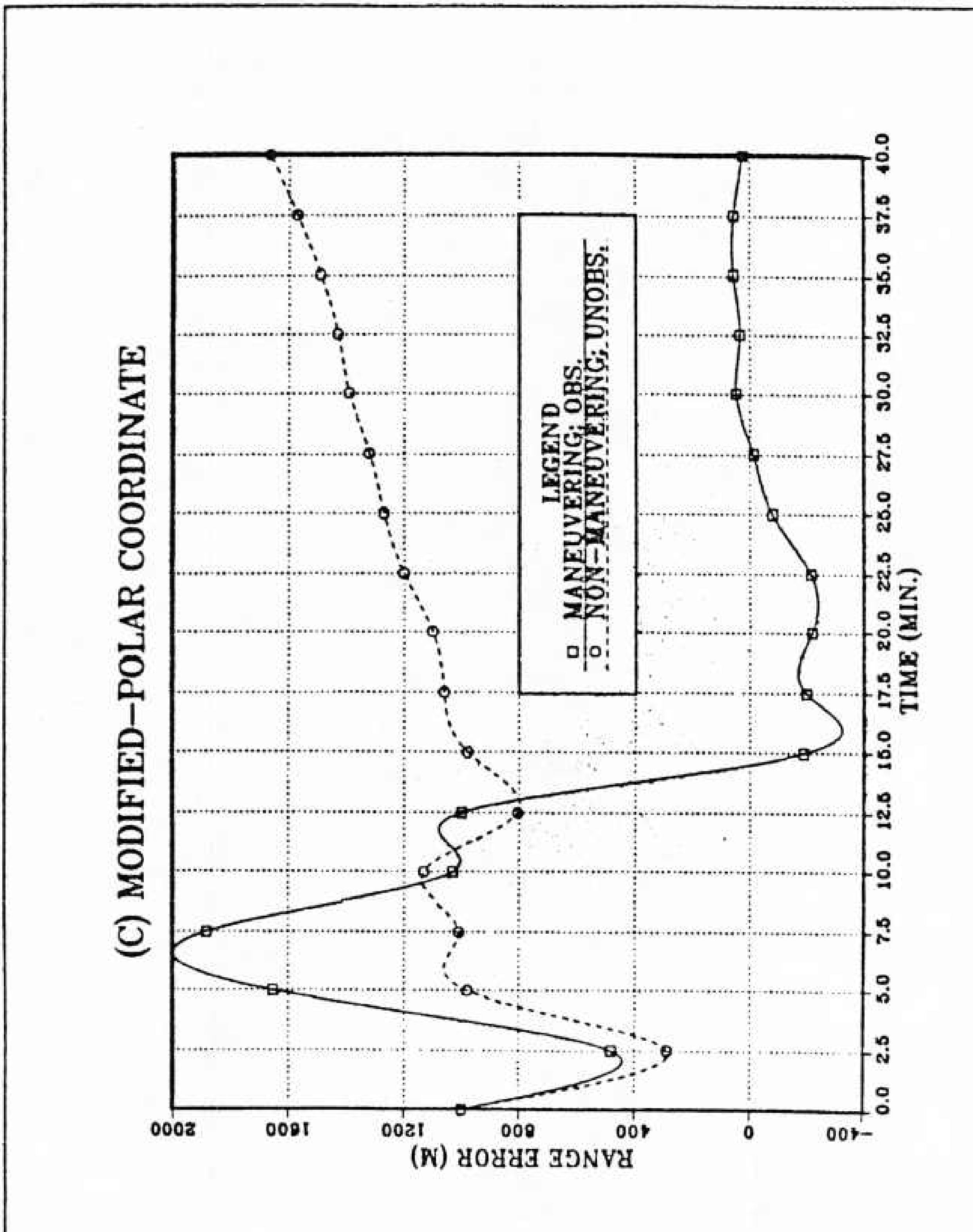


FIG 4,EFFECT OF NOISE LEVEL

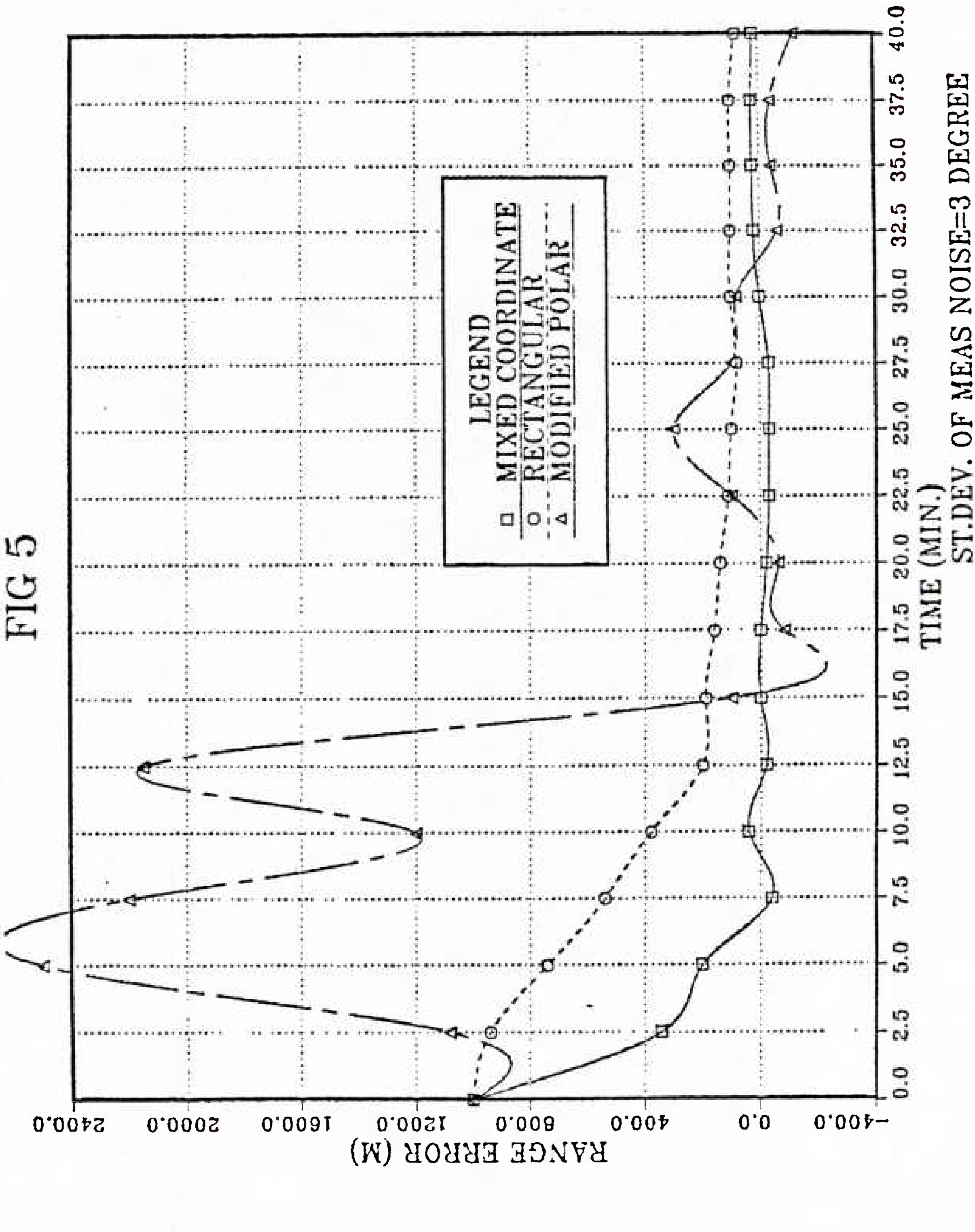


ST.DEV.OF MEAS NOISE=1 DEGREE



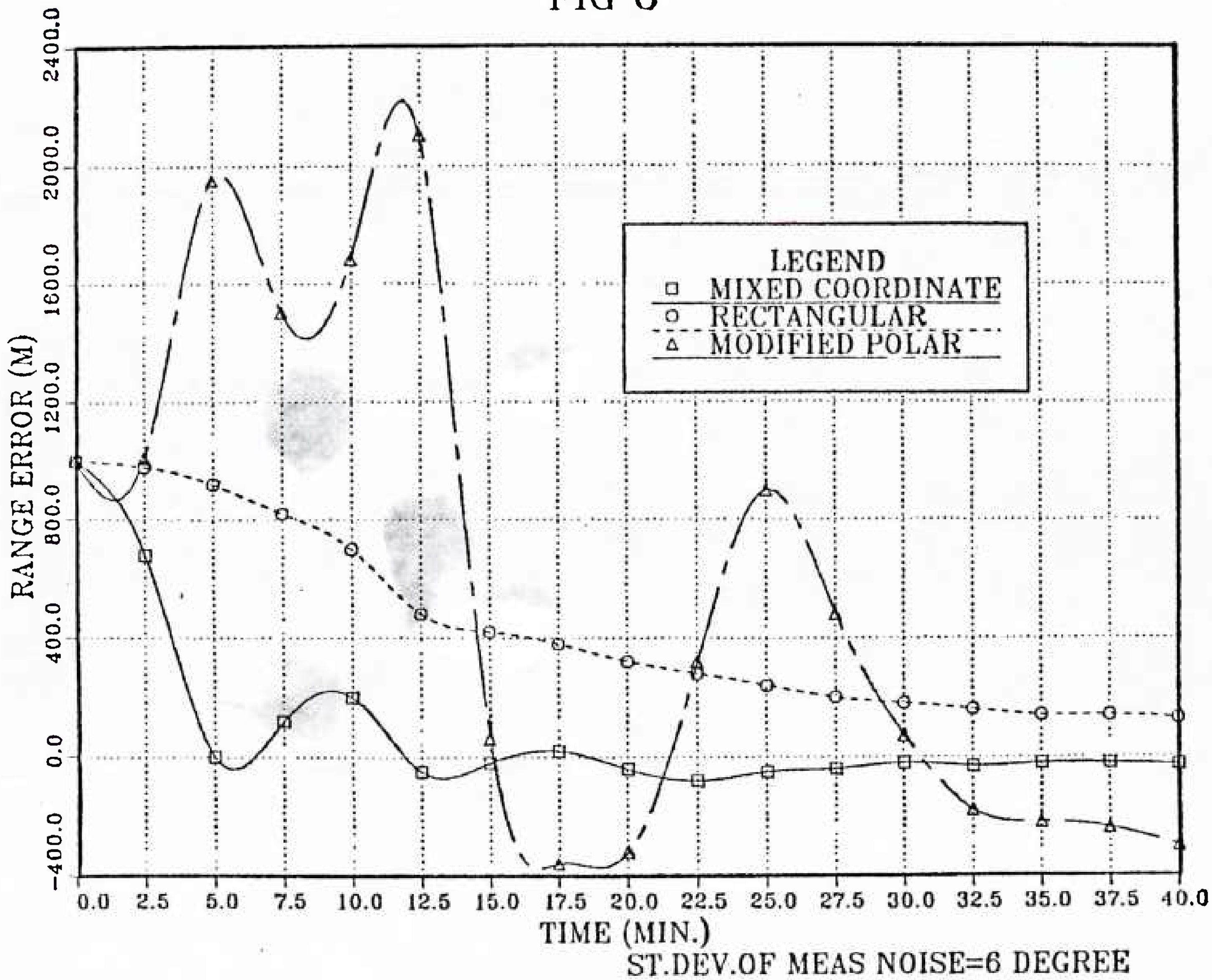
(C) MODIFIED–POLAR COORDINATE
LEGEND
MANEUVERING; OBS.
NON–MANEUVERING; UNOBS.
RANGE ERROR (M)
-400
0
400
800
1200
1600
2000
TIME (MIN.)
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
22.5
25.0
27.5
30.0
32.5
35.0
37.5
40.0

FIG 5



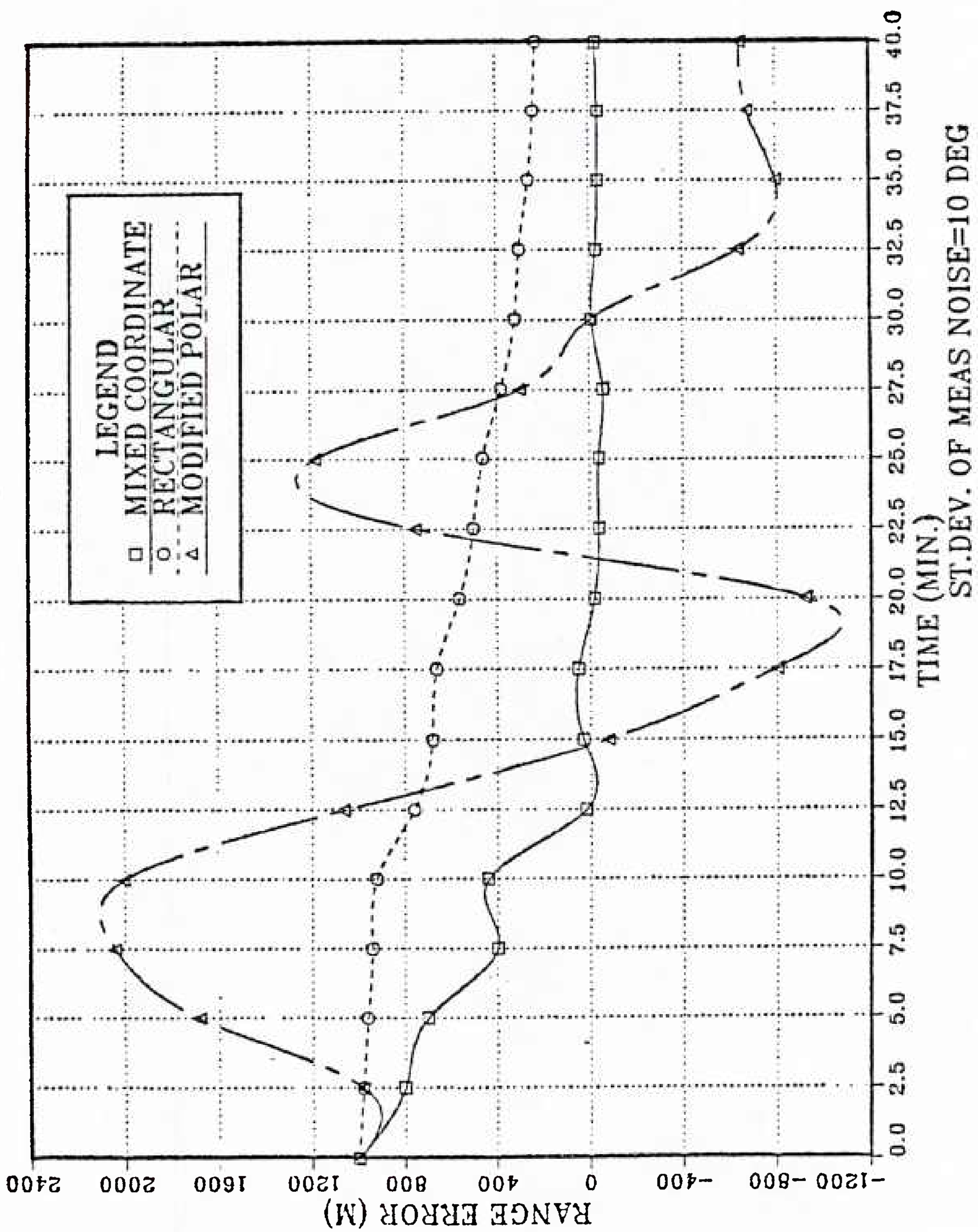
LEGEND
MIXED COORDINATE
RECTANGULAR
MODIFIED POLAR
RANGE ERROR (M)
-400.0
0.0
400.0
800.0
1200.0
1600.0
2000.0
2400.0
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
22.5
25.0
27.5
30.0
32.5
35.0
37.5
40.0
TIME (MIN.)
ST.DEV. OF MEAS NOISE=3 DEGREE

FIG 6



LEGEND
MIXED COORDINATE
RECTANGULAR
MODIFIED POLAR
RANGE ERROR (M)
2400.0
2000.0
1600.0
1200.0
800.0
400.0
0.0
-400.0
0.0 2.5 5.0 7.5 10.0 12.5 15.0 17.5 20.0 22.5 25.0 27.5 30.0 32.5 35.0 37.5 40.0
TIME (MIN.)
ST.DEV.OF MEAS NOISE=6 DEGREE

FIG 7



LEGEND
MIXED COORDINATE
RECTANGULAR
MODIFIED POLAR
RANGE ERROR (M)
2400
2000
1600
1200
800
400
0
-400
-800
-1200
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
22.5
25.0
27.5
30.0
32.5
35.0
37.5
40.0
TIME (MIN.)
ST.DEV. OF MEAS NOISE=10 DEG